2019-Sep-19 (pre-lecture)

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Previously: We proved earlier that we can find solutions to first-order autonomous ODEs $\dot{x} = f(x)$, demonstrating existence.

More than just existence, if $f(x_0) \neq 0$, we could explicitly write down the only local solution by defining

$$F(x) = \int_{x_0}^{x} \frac{dy}{f(y)}, \quad \text{and taking} \quad \phi(t) = F^{-1}(t).$$

This makes that solution locally unique.

If the interval of validity $(x_1, x_2) = \mathbb{R}$, then the solution is globally unique

Around some x_0 s.t. $f(x_0) = 0$, however, we could extend the solution in a non-unique way, only if

$$\left| \int_{x_0}^{x_0+\varepsilon} \frac{dy}{f(y)} \right| < \infty.$$

When $\left| \int_{x_0}^{x_0+\varepsilon} \frac{dy}{f(y)} \right| = \infty$ for all $x_0 \in \mathbb{R}$ s.t. $f(x_0) = 0$, we thus again have a globally unique solution.

(Hint: Teschl 1.10 bonus)

Can we say anything about general ODEs?

We are going to prove local existence and uniqueness for solutions to general ODEs.

Note: We are not going to solve general ODEs.
i.e., we still won't know explicitly the solution.

But we will show there exists a locally unique solution that we may or may not be able to find.

Key ingredient: Fixed point theorems and contractions.

Intuition:

Fixed point: Given an arbitrary mapping $f: X \to X$ any point x where $f(x) = x$.

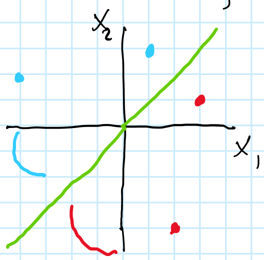
Ex. $X = \mathbb{R}$
$$\begin{cases} f(x) = x^2. & \text{Fixed points } 0 \text{ and } 1 \\ f(x) = \frac{x}{2}. & \text{Fixed points } 0. \end{cases}$$

$X = \mathbb{R}$: $f(x) = \frac{x}{2}$. Fixed points 0.

$f(x) = x+1$. No fixed points

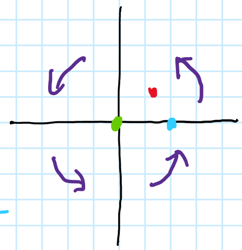
$X = \mathbb{R}^2$:

$f(x) = Ax$, $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $x \in \mathbb{R}$



Fixed points $x_1 = x_2$

$f(x) = Ax$, $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $\theta = 45^\circ$



Fixed point $(0,0)$.

$X = C^\infty(\mathbb{R})$ - i.e. let's consider fixed points in the space of infinitely differentiable functions on the real line

Let our mapping be $\frac{d}{dx}$.

What are the fixed points? $\frac{d}{dx} 0 = 0$

$$\frac{d}{dx} ce^x = ce^x$$

What about the mapping $K(f)$ defined by

$$K(f)(x) = 1 + \int_0^x f(y)\,dy \ ?$$

i.e. need $f(x) = 1 + \int_0^x f(y)\,dy$ (Integral Equation)

Try $f(x) = e^x$. $e^x = 1 + \int_0^x e^y\,dy = 1 + e^y\Big|_0^x = 1 + e^x - e^0 = e^x$.

So we can find fixed points for all kinds of mappings.

Contractions: A mapping that makes everything get closer.

Ex. $f(x) = \frac{x}{2}$. i.e. $|f(x) - f(y)| = \frac{1}{2}|x-y| \leq \theta |x-y|$, for some $\theta \in [0,1)$

Mentimeter: Are these contractions?

$$f(x) = x^2$$

$$f(x) = \sqrt{x}$$

$$f(x) = 1000 + \frac{x}{2}$$

$$f(x) = Ax, \quad A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$f(x) = Ax, \quad A = \begin{bmatrix} 0.5\cos\theta & -0.5\sin\theta \\ 0.9\sin\theta & 0.9\cos\theta \end{bmatrix}$$

$$\frac{d}{dx}, \quad \int \qquad ?$$

We haven't defined distance yet for functions.

Notice: When things always get closer, we hypothesized that there must be a fixed point.

Turns out we can show in a variety of settings that given a contraction, there exists a unique fixed point.

TODO:

- Define distance for functions
- Show that a certain mapping on functions is a contraction
- Show that that mapping has a fixed point that is the solution to an ODE.

More rigorously:

Let X be a real vector space. A norm (distance) on X is a map $\|\cdot\| : X \to [0, \infty)$ s.t.

(i) $\|0\| = 0$, $\|x\| > 0$ for $x \neq 0$.

(ii) $\|\alpha x\| = |\alpha| \|x\|$ for $\alpha \in \mathbb{R}$ and $x \in X$.

(iii) $\|x+y\| \leq \|x\| + \|y\|$ for $x, y \in X$ (triangle inequality)

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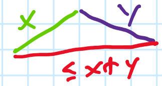

Together, $(X, \|\cdot\|)$ is a normed vector space.

A sequence of vectors x_n converges to f if

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

e.g. $x_n = \left(\frac{1}{10^n}, 1 - \frac{1}{10^n}\right) = \{(0.1, 0.9), (0.01, 0.99), \ldots\}$ converges to $(0,1)$.

A mapping $F: X \to Y$ between normed vector spaces is called continuous if $x_n \to x$ implies $F(x_n) \to F(x)$.

Note: The norm, vector addition, and multiplication by scalars are continuous
(Teschl 2.2)

When do sequence converge? When they get close to one another.

A sequence $x_1, x_2, \ldots$ is a Cauchy sequence if for any $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. for all $m, n > N$, $\|x_m - x_n\| < \varepsilon$.

A space is called complete if every Cauchy sequence has a limit.

A complete normed space is called a Banach space.

ex. $\mathbb{R}^n$ and $\mathbb{C}^n$ are Banach spaces (with Euclidean norm)

But what about function spaces?

Let $I \subseteq \mathbb{R}$ be a closed finite interval on the real line.
(compact)

Consider $C(I)$, the space of continuous functions on I.

Define $\|x\| = \sup_{t \in I} |x(t)|$.

Is this a metric? Check all three properties.

A sequence of functions $x_n(t)$ converges to $x(t)$ iff

$$\lim_{n\to\infty} \|x_n - x\| = \lim_{n\to\infty} \sup_{t\in I} |x_n(t) - x(t)| = 0$$

(uniform convergence)

Turns out (real analysis) that a Cauchy sequence

$\lim_{n\to\infty}$ … $\lim_{n\to\infty} \sup_{t\in I}$ … (convergence)

Turns out (real analysis) that a Cauchy sequence under this metric converges to another continuous function.

Thus, $C(I)$ with the metric $\|x\| = \sup_{t\in I} |x(t)|$ is a Banach space.

Next time: Formally define fixed points and contractions, which will allow us to prove existence and uniqueness of ODE solutions.